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GROUPS CONTAINING THE LARGEST POSSIBLE NUMBER OF OPERATORS OF ORDER TWO.

By G. A. MILLER.

The abelian group of order 2^m and of type (1, 1, 1, ...) contains 2^m-1 operators of order two. This group will not be considered in what follows. Let G represent any other group whose order (g) is divisible by the prime number p. We proceed to prove that the number of the operators of G whose orders exceed two is at least $\frac{p-1}{2p}g$ and to determine all the possible groups in which the number of these operators is exactly $\frac{p-1}{2p}g$. The considerations are extremely simple but the results seem of sufficient importance to be given in an explicit form.

If g is divisible by 2 but not by 4 it is well known that G contains a subgroup (H) of half its order which is composed of its operators of odd order in addition to the identity. In such a group the number of operators of order 2 cannot exceed $g/2$, and whenever the number of operators of order two contained in G is exactly $g/2$ then the order of G is twice an odd number and the subgroup H is abelian. These known facts are stated here merely for the sake of completeness. They are not essential in what follows.

If g is divisible by 4 it is clearly possible to construct a G such that more than half its operators are of order 2. Such a group is obtained by extending the cyclic group of order $g/2$ by the operators of order two which transform each operator of this cyclic group into its inverse. The group obtained in this manner is known as the dihedral rotation group of order g. In what follows it will be assumed that g is divisible by 4, and that more than half the operators of G are of order two. As it is assumed that G contains operators whose orders exceed two and as G is generated by its operators of order two, it follows that G is non-abelian and contains non-invariant operators of order 2.

Let t represent any non-invariant operator of order 2 contained in G. The subgroup (K) of G which is composed of all its operators which are commutative with t must include t. Let $G-K$ represent all the operators of G which are not also in K. All the products obtained by multiplying t into operators of $G-K$ are found in $G-K$. Since t is not commutative with any of these operators, the products obtained by multiplying t into any operator of order 2 in $G-K$ are of orders greater than 2. That is, $G-K$ contains at least as many operators whose orders exceed 2 as of order 2. Hence more than half the operators of K are of order 2.

If $G-K$ contains an operator of order p the index of K under G cannot be less than p.[*] In this case the number of operators whose orders exceed p in $G-K$ is not less than $\frac{p-1}{2p}g$ since the total number of operators in $G-K$ is not less than $\frac{p-1}{p}g$. If $G-K$ does not contain an operator of order p such an operator must occur in K as g is divisible by p. In this case, $G-K$ cannot contain less than $g/2-g/2l$ operators whose orders exceed 2, l being the index of K under G.

As the order of K is less than that of G and as K satisfies the conditions which G was assumed to satisfy whenever K includes an operator of order p,[†] we shall for the present suppose that the theorem in question is true with respect to K. That is, K contains at least $\frac{p-1}{2p}.g/l$ operators whose orders exceed two. In this case G must contain at least $g/2-g/2l+\frac{p-1}{2p}.g/l=\frac{pl-1}{2pl}g$ operators whose orders exceed two. As this number exceeds $\frac{p-1}{2p}g$, the theorem in question is true with respect to G provided it is true with respect to K. As it is evidently true with $4p$, it is universally true.

Having proved that the number of operators whose orders exceed 2 is at least equal to $\frac{p-1}{2p}g$, it remains to determine all the possible groups of order g in which there are exactly $\frac{p-1}{2p}g$ operators whose orders exceed two. We shall arrive at the remarkable result that *all such groups are the direct products of the dihedral rotation group of order $4p$ and an abelian group of order 2^α and of type* (1, 1, 1, ...). When $p>2$ such a group is also the direct product of the dihedral rotation group of order $2p$ and an abelian group of order $2^{\alpha-1}$ and of type (1, 1, 1, ...). We proceed to prove these results.

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[*] This follows from the usual rectangular arrangement of the operators of G, the operators of K constituting the first row of the rectangle. The theorem is included in the following known theorem: If a group G contains a subgroup K of index l, and if n is the lowest power of some operator (s) of G such that s^n is contained in K, then $n<l+1$.

[†] The order of K is divisible by 4 since it contains an invariant operator of order two and other operators of this order.

It has been proved above that K is the abelian group of order 2^β and of type (1, 1, 1, ...) whenever G contains a minimum number of operators whose orders exceed two. As this must be true of every K for the different non-invariant operators of order two contained in G and as the index of all of these K's is p, it follows that each non-invariant operator of order two contained in G has just p conjugates under G. Since any K is maximal it follows that the operators which are common to two K's are invariant under G, and vice versa. We shall soon see that these common operators include just half of all the operators of G.

As all the products obtained by multiplying t into operators whose orders exceed two are of order two, t must transform every operator whose order exceeds two into its inverse. Hence every non-invariant operator of order two contained in G must have this property. These operators constitute half the operators of $G-K$ and at least half the operators of K. Since they could not constitute more than half of the operators of G (for there are just as many operators that are commutative with any operator in G as there are of those which transform this operator into its inverse) it results that just half the operators of K are invariant under G. These operators constitute an invariant subgroup K' of order $g/2p$.

When $p>2$, G includes an invariant subgroup of order $2p$, since an operator of order p is transformed either into itself or into its inverse by all the operators of G. In this case G is the direct product of K' and this subgroup, the dihedral rotation group of order $2p$, since these two invariant subgroups have only the identity in common and the product of their orders is g*. When $p=2$; that is, when $g=2^m$, K is invariant.† Half of the operators of $G-K$ are of order 4. Hence G includes the octic group as an invariant subgroup. As this has two operators in common with K' such a G is the direct product of the octic group and any subgroup of half the order of K' and included in K' which does not include the commutator of order 2. This completes the proof of the theorems in question; and, as these direct products are so elementary, it seems unnecessary to go into further details.

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*Cf. Burnside, Theory of Groups of Finite Order, 1897, p. 44.

†In all other cases K has p conjugates under G.